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Transfer matrix eigenvalues of the anisotropic multiparametric U model

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Abstract

A multiparametric extension of the anisotropic U model is discussed which maintains integrability. The R -matrix solving the Yang–Baxter equation is obtained through a twisting construction applied to the underlying $U_q(sl(2|1))$ superalgebraic structure which introduces the additional free parameters that arise in the model. Three forms of Bethe ansatz solution for the transfer matrix eigenvalues are given which we show to be equivalent.

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1. Introduction

The quantum algebras, including the \mathbb{Z}_2 -graded analogues known as quantum superalgebras, play a central role in the construction of solutions of the Yang–Baxter equation which in turn may be used to construct integrable one-dimensional quantum models. Quantum algebras as defined originally by Jimbo and Drinfeld [1, 2] arise as one-parameter deformations of the familiar Lie algebras in such a way that the resulting algebraic structure is that of a quasi-triangular Hopf algebra. The supersymmetric generalizations are defined in [3–5] and the quasi-triangularity of these Hopf (super)algebras is discussed in [6]. The importance of this class of algebras is the existence of a universal element, known as the R -matrix, which gives a solution for the Yang–Baxter equation. For each given solution of the Yang–Baxter equation, there is a well known procedure called the quantum inverse scattering method (QISM) [7] by which an integrable one-dimensional quantum system is obtained. One of the key steps in the QISM is the construction of the transfer matrix which yields a family of mutually commuting operators (including the Hamiltonian) which are taken as constants of the motion for the system. Diagonalization of the transfer matrix is the main objective in the solution of the model and

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from this result many properties such as the ground state structure and elementary excitations can be deduced.

Through the construction of twisting by elements satisfying the two co-cycle condition it became apparent that new quasi-triangular Hopf algebras could be manufactured from existing ones [8], and if the twisting procedure introduced additional free parameters then these would appear in the Hopf algebra structure [9]. Viewed in another perspective, there exists classes of quasi-triangular Hopf algebras in which each member of the class is related to all others via twisting. In terms of representation theory, this means that representations of all quasi-triangular Hopf algebras in a given class are equivalent.

On the other hand, in applications to the area of constructing integrable quantum chain models some subtleties emerge [10]. In a recent work, it was shown that in the case of Reshetikhin twists [9], which will be defined precisely later, the periodic multiparametric chains can be mapped to the standard ones with the inclusion of a generalized twisted boundary condition [11]. In that work, two physical examples of the anisotropic t - J and U models of correlated electrons (both derived through representations of the quantum superalgebra $U_q(sl(2|1))$) were constructed and also the Bethe ansatz equations obtained. From these solutions, it is evident that the additional parameters introduced by twisting cannot be transformed away and thus do impact on the physics that these models describe.

Subsequent to the work of Reshetikhin [9], Engeldinger and Kempf [12] gave a more general prescription for the twisting element (or twistor) by relaxing the triangularity property. This latter construction opens the possibility for even more parameters to be incorporated into the model in an integrable fashion. When applied to the fundamental representation of the $U_q(gl(m|n))$ series, the method of Engeldinger and Kempf reproduces the construction of Reshetikhin. For this reason, we cannot use these more general techniques to obtain a more general extension of the anisotropic t - J model than that already obtained in [11]. However, when applied to ‘higher spin’ representations, such as the case to be considered here, the differences between the constructions begin to emerge. We will show that these additional parameters describe local basis transformations for the local Hamiltonians. This result will be illustrated for the anisotropic U model which will be discussed in some detail. Being a fermionic model, it is necessary to derive the model in a supersymmetric formulation of the QISM.

Unlike the case of integrable models based on non-graded algebras, there exist many examples of models with an underlying superalgebraic structure which admit more than one Bethe ansatz solution [13–20]. This facet was first discussed in [13] where it was shown that different choices for the chain of canonical subalgebra embeddings for $gl(m|n)$ used in the nested Bethe ansatz method leads to different solutions. An alternative point of view is that the non-uniqueness of solution stems from the fact that the definition for a Lie superalgebra in terms of a system of simple roots is not unique [21]. For the case of the supersymmetric t - J model two forms of solution were known long ago from the works of Lai [22] and Sutherland [23]. With the advent of the algebraic form of the Bethe ansatz a third form was proposed in [14, 15]. Moreover, Essler and Korepin showed further in [14] through an analytical argument that the three forms of solution for the supersymmetric t - J were equivalent.

For the isotropic U model, as a result of sharing the same supersymmetry algebra (namely, $sl(2|1)$) as the supersymmetric t - J model, there are also three forms of the Bethe ansatz solution. For the transfer matrix eigenvalues, two forms were obtained by Pfanmüller and Frahm [17] using standard methods. The third form was eventually also discovered by Pfanmüller and Frahm [18] but for this case one has to resort to more sophisticated Bethe ansatz techniques which were developed by Abad and Ríos [24].

Recall that the anisotropic U model was first introduced and solved via the co-ordinate

Bethe ansatz in [25] as a generalization of both the Bariev model [26] and the (isotropic) U model given in [27]. It was subsequently shown that the model is also obtained from an R -matrix solution of the Yang–Baxter equation obtained through the one-parameter family of four-dimensional (minimal typical) representations of the quantum superalgebra $U_q(sl(2|1))$ [28] (see also [29–32]). By applying the twisting construction to this solution of the Yang–Baxter equation we will give the explicit form for the resulting R -matrix and in turn derive a multiparametric generalization of the anisotropic U model.

One of the main objectives of this paper is to present all three forms of the Bethe ansatz solution for the multiparametric anisotropic U model including the Bethe ansatz equations and explicit eigenvalue expressions for the transfer matrix. Since the model is not based on the fundamental representation of its supersymmetry algebra, we will adopt a generalization of the procedure used in the case of the solution of the integrable spin 1 chain (Fateev–Zamolodchikov model) [33–35]. In the model considered here this involves determining the L -operator which acts on the mixed tensor space of the fundamental module with the four-dimensional module which describe the local quantum states. A similar procedure has recently been undertaken by Grunberg [36, 37] for classes of $U_q(sl(2|1))$ models which include the usual anisotropic U model. The necessary step for the multiparametric case is the determination of the appropriate multiparametric L -operator. In principle, it may be possible to work directly with the R -matrix for the Bethe ansatz solution in an analogous way to that used by Ramos and Martins in the isotropic case [38] or even the analytic Bethe ansatz approach developed by Tsuboi for quantum superalgebras [39]. However, we will not consider these options in this paper.

Once the three forms of the Bethe ansatz solution have been obtained, we will proceed to argue that the transfer matrix eigenvalues are equivalent. Our approach is somewhat different than that used by Essler and Korepin for the t – J model [14]. We use the result that the Bethe ansatz solution for a $U_q(sl(1|1))$ (free fermion) model can be approached in two different ways and then proceed to show that the three solutions for the U model have their origin in the differing forms of these $U_q(sl(1|1))$ eigenvalue expressions.

One of the sequent results of the Bethe analysis for the transfer matrix eigenvalues is that the eigenvalues of the *quantum* transfer matrix are obtained with little additional effort. The quantum transfer matrix approach, which has primarily been developed by Klümper and collaborators [40–42], has proved to be a powerful method to determine thermodynamic properties of an integrable model at finite temperature. In the final section of this paper we will present the eigenvalues of the quantum transfer matrix for the anisotropic multiparametric U model.

2. Quantum inverse scattering method

We begin by reviewing the fundamental features of the QISM. Let $R(u) \in \text{End } V \otimes V$ be a solution of the Yang–Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \quad (1)$$

For full generality, we consider the cases also when V denotes a \mathbb{Z}_2 -graded vector space. In such instances it is necessary to impose the following rule for the tensor product multiplication of matrices:

$$(a \otimes b)(c \otimes d) = (-1)^{[b][c]}ac \otimes bd \quad (2)$$

for matrices a, b, c, d of homogeneous degree. The symbol $[a] \in \mathbb{Z}_2$ denotes the degree of the matrix a . The monodromy matrix is defined

$$\mathcal{T}(u) = R_{0L}(u)R_{0(L-1)}(u) \dots R_{01}(u)$$

from which the transfer matrix is given by

$$t(u) = \text{str}_0 \mathcal{T}(u). \quad (3)$$

A consequence of (1) is

$$[t(u), t(v)] = 0 \quad \forall u, v \in \mathbb{C}.$$

Above str_0 denotes the supertrace taken over the auxiliary space which is labelled by 0.

In the usual manner the Hamiltonian associated with the transfer matrix is defined by the relation

$$H = t^{-1}(u) \left. \frac{d}{du} t(u) \right|_{u=0}.$$

Assuming regularity of the R -matrix; i.e.

$$R(0) = P$$

where P is the (\mathbb{Z}_2 -graded) permutation operator, yields

$$H = \sum_{i=1}^{L-1} h_{i(i+1)} + h_{L1}$$

where the local two site Hamiltonians are given by

$$h = \left. \frac{d}{du} P R(u) \right|_{u=0}.$$

Another observable operator which is readily obtained from the QISM is the momentum operator which is given by [7]

$$p = i \ln t(0). \quad (4)$$

It is more useful for our purposes to work with the exponentiated form

$$\begin{aligned} T &= \exp(-ip) \\ &= t(0) \\ &= P_{1L} \dots P_{13} P_{12} \end{aligned}$$

which we call the translation operator. It satisfies the relations

$$T h_{i(i+1)} T^{-1} = h_{(i+1)(i+2)} \quad T h_{L1} T^{-1} = h_{12}.$$

Clearly

$$[T, H] = 0$$

which reflects translational invariance of the periodic model.

By using an available construction [9] for obtaining multiparametric quantum algebras, it is straightforward to obtain the associated multiparametric quantum spin chain. Below we will describe this construction and show that in each case we can effectively map the additional parameters to a ‘generalized boundary condition’.

3. Reshetikhin twists

Let (A, Δ, R) denote a quasitriangular Hopf (super)algebra where Δ and R denote the co-product and R -matrix respectively. Suppose that there exists an element $F \in A \otimes A$ such that

$$\begin{aligned} (\Delta \otimes I)(F) &= F_{13}F_{23} \\ (I \otimes \Delta)(F) &= F_{13}F_{12} \\ F_{12}F_{13}F_{23} &= F_{23}F_{13}F_{12}. \end{aligned} \tag{5}$$

Then (A, Δ^F, R^F) is also a quasitriangular Hopf (super)algebra with co-product and R -matrix respectively given by

$$\Delta^F = F_{12}\Delta F_{12}^{-1} \quad R^F = F_{21}R F_{12}^{-1}. \tag{6}$$

Throughout we refer to F as a *twistor*.

The result stated above is a little more general than that originally proposed by Reshetikhin and is due to Engeldinger and Kempf [12]. In the original work [9] Reshetikhin imposed the additional constraint (triangularity property)

$$F_{12}F_{21} = I \otimes I$$

and in the case that (A, Δ, R) is an affine quantum (super)algebra Reshetikhin gave the example that F can be chosen to be

$$F = \exp \sum_{i < j} (H_i \otimes H_j - H_j \otimes H_i) \phi_{ij} \tag{7}$$

where $\{H_i\}$ is a basis for the Cartan subalgebra of the affine quantum (super)algebra and the ϕ_{ij} , $i < j$ are arbitrary complex parameters. However, following the construction of Engeldinger and Kempf it is possible to choose

$$F = \exp \sum_{i,j} (H_i \otimes H_j) \phi_{ij} \tag{8}$$

which obviously gives a twistor dependent on more free parameters. Note that it is also possible to extend the Cartan subalgebra by an additional central extension (not the usual central charge) H_0 which will act as a scalar multiple of the identity operator in any irreducible representation.

Suppose that π is a loop representation of the affine quantum superalgebra. We let $R(u)$, $R^F(u)$ be the (super)matrix representatives of R and R^F respectively, which both satisfy the Yang–Baxter equation (1). As $R(0) = P$ then $R^F(0) = P$ as a result of (6). We may construct the transfer matrix

$$\begin{aligned} t^F(u) &= \text{str}_0(\pi^{\otimes(L+1)}(I \otimes \Delta_L^F)R_{01}^F) \\ &= \text{str}_0(R_{0L}^F(u)R_{0(L-1)}^F(u) \dots R_{01}^F(u)) \end{aligned} \tag{9}$$

where Δ_L^F is defined recursively through

$$\begin{aligned} \Delta_L^F &= (I \otimes I \dots \otimes \Delta^F)\Delta_{L-1}^F \\ &= (\Delta^F \otimes I \dots \otimes I)\Delta_{L-1}^F. \end{aligned} \tag{10}$$

Again the subscripts 0 and 1, 2, ..., L denote the auxiliary and quantum spaces respectively, and str_0 is the supertrace over the zeroth space. From the Yang–Baxter equation it follows that the multiparametric transfer matrices $t^F(u)$ form a commuting family. The associated multiparametric spin chain Hamiltonian is given by

$$\begin{aligned} H^F &= (t^F(u))^{-1} \frac{d}{du} t^F(u) \Big|_{u=0} \\ &= \sum_{i=1}^{L-1} h_{i,i+1}^F + h_{L1}^F \end{aligned} \tag{11}$$

with

$$\begin{aligned} h^F &= \left. \frac{d}{du} P R^F(u) \right|_{u=0} \\ &= F h F^{-1}. \end{aligned} \quad (12)$$

The above construction allows us a means to incorporate arbitrary parameters (through the ϕ_{ij}) into the Hamiltonian without corrupting integrability. We will refer to such extra variables in the model as *gauge parameters*.

Through use of (5) we may alternatively write

$$t^F(u) = \text{str}_0(\pi^{\otimes(L+1)}(I \otimes J_L)[(I \otimes \Delta_L)(F_{10} R_{01} F_{10})](I \otimes J_L)^{-1})$$

with

$$\begin{aligned} J_L &= G_{L-1} G_{L-2} \dots G_1 \\ G_i &= F_{iL} F_{i(L-1)} \dots F_{i(i+1)}. \end{aligned} \quad (13)$$

We now define a new transfer matrix

$$\begin{aligned} t(u) &= J_L^{-1} t^F(u) J_L \\ &= \text{str}_0(\pi^{\otimes(L+1)}(I \otimes \Delta_L)(F_{10} R_{01} F_{01}^{-1})) \end{aligned} \quad (14)$$

where we have employed the convention to let F denote both the algebraic object and its (super)matrix representative. Through further use of (5) we may show that

$$t(u) = \text{str}_0(F_{10} F_{20} \dots F_{L0} R_{0L}(u) R_{0(L-1)}(u) \dots R_{01}(u) F_{01}^{-1} \dots F_{0L}^{-1}).$$

In this setting the operator T assumes the form

$$T = F_{12}^{-1} F_{13}^{-1} \dots F_{1L}^{-1} F_{21} F_{31} \dots F_{L1} P_{1L} \dots P_{13} P_{12}$$

and the associated Hamiltonian is given by

$$\begin{aligned} H &= t^{-1}(u) \left. \frac{d}{du} t(u) \right|_{u=0} \\ &= \sum_{i=1}^{L-1} h_{i,i+1} + h_b \end{aligned} \quad (15)$$

where

$$h_b = F_{(L-1)L}^{-1} \dots F_{1L}^{-1} F_{L(L-1)} \dots F_{L1} h_{L1} F_{L1}^{-1} \dots F_{(L-1)L}^{-1} F_{1L} \dots F_{(L-1)L}.$$

It is thus apparent that the matrix J_L transforms the multiparametric Hamiltonian to one where the parameters appear only in a generalized boundary term h_b . Although the boundary term is transformed by a global operator, there are some important properties which should be noted that show that the generalized boundary interaction behaves as a two-site operator. The first is that

$$[h_b, h_{i(i+1)}] = 0 \quad \text{for } i \neq 1, L-1. \quad (16)$$

Thus if we think of the local Hamiltonians as observables then we can still independently measure the boundary two-site energies and those within the bulk. In addition, translational invariance is maintained; namely

$$[T, H] = 0$$

and more importantly

$$\begin{aligned} T h_b T^{-1} &= h_{12} & T h_{(L-1)L} T^{-1} &= h_b \\ T h_{i(i+1)} T^{-1} &= h_{(i+1)(i+2)} & \text{for } i &\neq L-1. \end{aligned} \quad (17)$$

Consequently, such a model can still be interpreted as describing a closed chain system. This situation bears close similarity with the closed quantum superalgebra invariant models [43].

4. Jacobs–Cornwell twists

More recently, a new type of twisting 2-cocycle has been introduced by Jacobs and Cornwell [44] in their work on relating non-standard quantum algebras to standard ones. In notation as above, suppose there exists $F \in A \otimes A$ which satisfies the following relations:

$$\begin{aligned} F_{12}F_{23} &= F_{23}F_{12} \\ (\Delta \otimes I)F &= F_{23}F_{13} \\ (I \otimes \Delta)F &= F_{12}F_{13}. \end{aligned}$$

Then (A, Δ^F, R^F) is also a quasi-triangular Hopf algebra with Δ^F, R^F given by equation (6) above.

As in the case of the Reshetikhin twists above, one may use the Jacobs–Cornwell twists to construct multiparametric chains which can also be mapped to a system where the extra parameters occur only in a generalized boundary interaction. For these models the transformation takes the form

$$\begin{aligned} J_L &= G_{L-1} \dots G_2 G_1 \\ G_i &= F_{i(i+1)} F_{i(i+2)} \dots F_{iL}. \end{aligned}$$

The translation operator under this mapping becomes

$$T = F_{1L}^{-1} F_{1(L-1)}^{-1} \dots F_{12}^{-1} F_{L1} \dots F_{21} P_{1L} \dots P_{13} P_{12}$$

while the generalized boundary term h_b in the Hamiltonian is

$$h_b = F_{1L}^{-1} \dots F_{(L-1)L}^{-1} F_{L1} \dots F_{L(L-1)} h_{L1} F_{L(L-1)}^{-1} \dots F_{L1}^{-1} F_{(L-1)L} \dots F_{1L}.$$

The properties (16), (17) also hold for the Jacobs–Cornwell twists.

Finally, it is important to mention that the twistor equation (8) qualifies as both a Jacobs–Cornwell twist as well as a Reshetikhin twist. In general however, these two classes of twistors are inequivalent (see [44]).

5. Symmetric twists

As mentioned earlier, the antisymmetrization condition equation (7) originally imposed by Reshetikhin can be relaxed and we can consider the more general Cartan subalgebra twists of the form equation (8) which introduce more free parameters into the Hamiltonian. We will show here, however, that these additional parameters simply describe local basis transformations and as such do not have a bearing on the spectrum of the model.

We begin with the observation that all the twistors of the type equation (8) close to form a commutative group. Moreover, each twistor can be expressed as a product of a symmetric and antisymmetric twistor through

$$F = F^s \cdot F^a$$

where

$$F^s = (F_{12} \cdot F_{21})^{1/2} \quad F^a = (F_{12} \cdot F_{21}^{-1})^{1/2}.$$

The square root in the above expression is well defined given that the twistors of the form equation (8) are defined in terms of the exponential of an operator. Given two Cartan elements k, l , let us consider a symmetric twist given simply by

$$F = \exp(k \otimes l + l \otimes k). \quad (18)$$

It is a straightforward exercise to show that

$$F = \exp(\Delta(k \cdot l)) \cdot U_1 \cdot U_2$$

where $U = \exp(-k \cdot l)$ and consequently for a twistor of the type equation (18) we simply have

$$H^F = U_1 U_2 \dots U_L H U_1^{-1} U_2^{-1} \dots U_L^{-1}. \quad (19)$$

The fact that all symmetric twists arise as products of twistors of the form equation (18) allows us to conclude that the relation equation (19) is true for any symmetric twistor. As a result, we cannot expect that the Engeldinger–Kempf form of twistor equation (8) will introduce more *dynamical* gauge parameters than twistors of the Reshetikhin form equation (7).

6. The quantum transfer matrix method

The algebraic Bethe ansatz method for the solution of the multiparametric anisotropic U model which will be employed below also has an important application in the quantum transfer matrix method [40–42] which we will briefly describe here.

As a result of the previous discussions, we can conclude that the QISM allows us to determine that the relation between the Hamiltonian and the transfer matrix is of the general form

$$t(u) = T \exp[uH + o(u^2)].$$

Let us also define

$$\bar{t}(u) = \text{str}_0(R_{10}(u)R_{20}(u) \dots R_{L0}(u)).$$

A similar calculation shows that we may write

$$\bar{t}(u) = T^{-1} \exp[uH + o(u^2)]$$

and thus

$$(t(-\beta/\mathcal{L})\bar{t}(-\beta/\mathcal{L}))^{\mathcal{L}/2} = \exp[-\beta H + o(1/\mathcal{L})]$$

where $\beta = 1/kT$ and throughout we will assume that \mathcal{L} is even. Let us define two quantities U and \bar{U} through the relations

$$\begin{aligned} U &= \text{tr} (t(-\beta/\mathcal{L})\bar{t}(-\beta/\mathcal{L}))^{\mathcal{L}/2} \\ \bar{U} &= \text{str} (t(-\beta/\mathcal{L})\bar{t}(-\beta/\mathcal{L}))^{\mathcal{L}/2}. \end{aligned}$$

Both U and \bar{U} can be thought of as partition functions for a classical two-dimensional lattice model on a torus which differ only in boundary conditions. In the thermodynamic limit $\mathcal{L} \rightarrow \infty$ we can neglect such a difference in the boundary conditions and conclude that

$$\lim_{\mathcal{L} \rightarrow \infty} \bar{U} = \lim_{\mathcal{L} \rightarrow \infty} U = Z$$

where Z is the partition function for the quantum system with Hamiltonian H (derived from the QISM) in the thermodynamic limit.

At this point we define the quantum transfer matrix with inhomogeneity χ to be

$$Q(u) = \text{str}_0(R_{\mathcal{L}0}^{\text{st}\mathcal{L}}(\chi - u)R_{0(\mathcal{L}-1)}(\chi + u) \dots R_{20}^{\text{st}2}(\chi - u)R_{01}(\chi + u))$$

which forms a commuting family $[Q(u), Q(v)] = 0$ by virtue of the fact that the Yang–Baxter equation (1) is expressible in the equivalent form

$$R_{12}(u - v)R_{31}^{\text{st}3}(-u)R_{32}^{\text{st}3}(-v) = R_{32}^{\text{st}3}(-v)R_{31}^{\text{st}3}(-u)R_{12}(u - v)$$

where throughout st_i refers to the supertransposition taken over the i th space. Note that we may write

$$t(u) = \text{str}_0(R_{01}^{\text{st}_0}(u)R_{02}^{\text{st}_0}(u)\dots R_{0L}^{\text{st}_0}(u)).$$

The quantum transfer matrix allows us to now express Z as

$$Z = \lim_{\mathcal{L} \rightarrow \infty} \text{str}(Q(0)^{\mathcal{L}})$$

with $\chi = -\beta/\mathcal{L}$. The thermodynamical properties are determined by the maximum eigenvalue of $Q(u)$. Given that $Q(u)$ forms a commuting family, the traditional Bethe ansatz methods can be applied for the diagonalization.

7. Multiparametric anisotropic U model

As an application of the above formalism, here we will introduce the multiparametric anisotropic U model derived from an R -matrix obtained from the quantum superalgebra $U_q(\mathfrak{sl}(2|1))$. This superalgebra has simple generators $\{e_0, f_0, h_0, e_1, f_1, h_1\}$ corresponding to the simple roots associated with the Cartan matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.$$

It is worthwhile mentioning here that we work in the standard root system of one bosonic and one fermionic simple root. Another system of simple roots exists which we will not consider (see e.g. [32]). For a full definition of the algebra we refer to [30].

This algebra admits a non-trivial one parameter family of four-dimensional representations, which we label π , given by

$$\begin{aligned} \pi(e_0) &= \sqrt{[\alpha]}e_2^1 + \sqrt{[\alpha+1]}e_4^3 \\ \pi(f_0) &= \sqrt{[\alpha]}e_1^2 + \sqrt{[\alpha+1]}e_3^4 \\ \pi(h_0) &= \alpha(e_1^1 + e_2^2) + (\alpha+1)(e_3^3 + e_4^4) \\ \pi(e_1) &= e_3^2 \\ \pi(f_1) &= e_2^3 \\ \pi(h_1) &= e_2^2 - e_3^3. \end{aligned} \tag{20}$$

Above the indices of the elementary matrices e_j^i carry the \mathbb{Z}_2 -grading $(1) = (4) = 0$, $(2) = (3) = 1$ and we employ the notation

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

Throughout we work under the assumption that q is generic. Associated with this representation there is a solution of the Yang–Baxter equation which is obtained by solving Jimbo’s equations. The problem of obtaining this solution has been considered in [28–32]. Applying the construction we described in section 3, with the Cartan subalgebra extended by a central

element, yields the following multiparametric solution of (1)

$$R(u) = \begin{pmatrix} R_{11}^{11} & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & R_{12}^{12} & 0 & 0 & | & R_{21}^{12} & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & R_{13}^{13} & 0 & | & 0 & 0 & 0 & 0 & | & R_{31}^{13} & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{14}^{14} & | & 0 & 0 & R_{23}^{14} & 0 & | & 0 & R_{32}^{14} & 0 & 0 & | & R_{41}^{14} & 0 & 0 & 0 \\ - & - & - & - & | & - & - & - & - & | & - & - & - & - & | & - & - & - & - \\ 0 & R_{12}^{21} & 0 & 0 & | & R_{21}^{21} & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & R_{22}^{22} & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{23}^{23} & | & 0 & 0 & R_{23}^{23} & 0 & | & 0 & R_{33}^{23} & 0 & 0 & | & R_{41}^{23} & 0 & 0 & 0 \\ - & - & - & - & | & - & - & - & - & | & - & - & - & - & | & - & - & - & - \\ 0 & 0 & R_{13}^{31} & 0 & | & 0 & 0 & 0 & 0 & | & R_{31}^{31} & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{14}^{32} & | & 0 & 0 & R_{23}^{32} & 0 & | & 0 & R_{32}^{32} & 0 & 0 & | & R_{41}^{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & R_{33}^{33} & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & R_{34}^{34} & | & 0 & 0 & R_{43}^{34} & 0 \\ - & - & - & - & | & - & - & - & - & | & - & - & - & - & | & - & - & - & - \\ 0 & 0 & 0 & R_{14}^{41} & | & 0 & 0 & R_{23}^{41} & 0 & | & 0 & R_{32}^{41} & 0 & 0 & | & R_{41}^{41} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & R_{24}^{42} & | & 0 & 0 & 0 & 0 & | & 0 & R_{42}^{42} & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & R_{34}^{43} & | & 0 & 0 & R_{43}^{43} & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & R_{44}^{44} \end{pmatrix}$$

where the non-zero entries are given by

$$\begin{aligned} R_{11}^{11} &= \frac{[u-\alpha]}{[u+\alpha]} & R_{12}^{12} &= p_1 \frac{[u]}{[u+\alpha]} \\ R_{13}^{13} &= p_2 \frac{[u]}{[u+\alpha]} & R_{14}^{14} &= p_1 p_2 \frac{[u][u-1]}{[u+\alpha][u-\alpha-1]} \\ *R_{21}^{12} &= q^u \frac{[\alpha]}{[u+\alpha]} & *R_{31}^{13} &= q^u \frac{[\alpha]}{[u+\alpha]} \\ R_{41}^{14} &= q^{2u} \frac{[\alpha][\alpha+1]}{[u+\alpha][u-\alpha-1]} & *R_{23}^{14} &= p_2 p_3^{-1} q^{u-1/2} \frac{[\alpha]^{1/2}[\alpha+1]^{1/2}[u]}{[u+\alpha][u-\alpha-1]} \\ *R_{32}^{14} &= -p_1 p_4^{-1} q^{u+1/2} \frac{[\alpha]^{1/2}[\alpha+1]^{1/2}[u]}{[u+\alpha][u-\alpha-1]} & R_{21}^{21} &= p_1^{-1} \frac{[u]}{[u+\alpha]} \\ *R_{22}^{22} &= 1 & *R_{23}^{23} &= p_1^{-1} p_2 p_3^{-1} p_4 \frac{[u]^2}{[u+\alpha][u-\alpha-1]} \\ R_{24}^{24} &= p_2 p_3^{-1} p_4 \frac{[u]}{[u-\alpha-1]} & R_{12}^{21} &= -q^{-u} \frac{[\alpha]}{[u+\alpha]} \\ *R_{32}^{23} &= \frac{2q - q^{2\alpha+1} - q^{-2\alpha-1} - q^{2u+1} + q^{2u-1}}{(q^{u+\alpha} - q^{-u-\alpha})(q^{u-\alpha-1} - q^{-u+\alpha+1})} & R_{42}^{24} &= q^u \frac{[\alpha+1]}{[u-\alpha-1]} \\ *R_{14}^{23} &= -p_2 p_4 q^{-u+1/2} \frac{[\alpha]^{1/2}[\alpha+1]^{1/2}[u]}{[u+\alpha][u-\alpha-1]} & *R_{41}^{23} &= -p_1^{-1} p_4 q^{u+1/2} \frac{[\alpha]^{1/2}[\alpha+1]^{1/2}[u]}{[u+\alpha][u-\alpha-1]} \\ R_{31}^{31} &= p_2^{-1} \frac{[u]}{[u+\alpha]} & *R_{32}^{32} &= p_1 p_2^{-1} p_3 p_4^{-1} \frac{[u]^2}{[u+\alpha][u-\alpha-1]} \\ *R_{33}^{33} &= 1 & R_{34}^{34} &= p_1 p_3 p_4^{-1} \frac{[u]}{[u-\alpha-1]} \\ R_{13}^{31} &= -q^{-u} \frac{[\alpha]}{[u+\alpha]} & *R_{23}^{32} &= \frac{2q^{-1} - q^{2\alpha+1} - q^{-2\alpha-1} + q^{-2u+1} - q^{-2u-1}}{(q^{u+\alpha} - q^{-u-\alpha})(q^{u-\alpha-1} - q^{-u+\alpha+1})} \\ R_{43}^{34} &= q^u \frac{[\alpha+1]}{[u-\alpha-1]} & *R_{14}^{32} &= p_1 p_3 q^{-u-1/2} \frac{[\alpha]^{1/2}[\alpha+1]^{1/2}[u]}{[u+\alpha][u-\alpha-1]} \\ *R_{41}^{32} &= p_2^{-1} p_3 q^{u-1/2} \frac{[\alpha]^{1/2}[\alpha+1]^{1/2}[u]}{[u+\alpha][u-\alpha-1]} & R_{41}^{41} &= p_1^{-1} p_2^{-1} \frac{[u][u-1]}{[u+\alpha][u-\alpha-1]} \\ R_{42}^{42} &= p_2^{-1} p_3 p_4^{-1} \frac{[u]}{[u-\alpha-1]} & R_{43}^{43} &= p_1^{-1} p_3^{-1} p_4 \frac{[u]}{[u-\alpha-1]} \\ R_{44}^{44} &= \frac{[u+\alpha+1]}{[u-\alpha-1]} & R_{14}^{41} &= q^{-2u} \frac{[\alpha][\alpha+1]}{[u+\alpha][u-\alpha-1]} \\ *R_{24}^{42} &= -q^{-u} \frac{[\alpha+1]}{[u-\alpha-1]} & *R_{34}^{43} &= -q^{-u} \frac{[\alpha+1]}{[u-\alpha-1]} \\ *R_{23}^{41} &= p_1^{-1} p_3^{-1} q^{-u-1/2} \frac{[\alpha]^{1/2}[\alpha+1]^{1/2}[u]}{[u+\alpha][u-\alpha-1]} & *R_{32}^{41} &= -p_2^{-1} p_4^{-1} q^{-u+1/2} \frac{[\alpha]^{1/2}[\alpha+1]^{1/2}[u]}{[u+\alpha][u-\alpha-1]} \end{aligned} \tag{21}$$

We remind the reader that the above R -matrix solves the Yang–Baxter equation (1) subject to the rule equation (2) which is a consequence of the $U_q(sl(2|1))$ superalgebraic structure underlying this solution. The significance of the $*$ notation prefixing some matrix elements will be explained in a subsequent section.

Using the above solution, we can employ the QISM to obtain an integrable Hamiltonian as discussed earlier. Identifying the four basis states in the $U_q(sl(2|1))$ representation space

V with the local electronic states through

$$v^4 = |0\rangle, \quad v^3 = |\downarrow\rangle, \quad v^2 = |\uparrow\rangle, \quad v^1 = |\uparrow\downarrow\rangle \quad (22)$$

allows us to express the local Hamiltonians in the following form (with convenient normalization):

$$\begin{aligned} h_{i(i+1)} = & -p_2^{-1} p_3 p_4^{-1} c_{i\uparrow}^\dagger c_{(i+1)\uparrow} (q^{-1} p_4^2 [\alpha + 1] [\alpha]^{-1})^{\frac{1}{2}n_{i\downarrow}} (q p_3^{-2} [\alpha + 1] [\alpha]^{-1})^{\frac{1}{2}n_{(i+1)\downarrow}} + \text{h.c.} \\ & - p_1^{-1} p_3^{-1} p_4 c_{i\downarrow}^\dagger c_{(i+1)\downarrow} (q p_3^2 [\alpha + 1] [\alpha]^{-1})^{\frac{1}{2}n_{i\uparrow}} \\ & \times (q^{-1} p_4^{-2} [\alpha + 1] [\alpha]^{-1})^{\frac{1}{2}n_{(i+1)\uparrow}} + \text{h.c.} \\ & + [\alpha]^{-1} (p_1^{-1} p_2^{-1} c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger c_{(i+1)\downarrow} c_{(i+1)\uparrow} + \text{h.c.}) + [\alpha]^{-1} (n_{i\uparrow} n_{i\downarrow} + n_{i\uparrow} n_{i\downarrow}) \\ & + q^{\alpha+1} (n_{i\uparrow} + n_{i\downarrow} - 1) + q^{-\alpha-1} (n_{(i+1)\uparrow} + n_{(i+1)\downarrow} - 1). \end{aligned} \quad (23)$$

Above we have used standard notation; the operators c^\dagger , c denote fermi creation and annihilation operators and n measures occupation number. For the above operator to be hermitian, it is assumed that the parameters p_j , $j = 1, 2, 3, 4$ lie on the unit circle. This results from the fact that in the h.c. terms one has $p_j \rightarrow p_j^{-1}$.

An immediate feature of this Hamiltonian is that it is not spin reflection invariant. However, it is invariant with respect to spin reflection coupled with the interchange of parameters

$$q \leftrightarrow q^{-1} \quad p_1 \leftrightarrow p_2 \quad p_3 \leftrightarrow p_4. \quad (24)$$

This invariance will manifest itself in the Bethe ansatz solutions determined later.

Although the above local Hamiltonian depends on the four parameters p_j , we can apply the unitary transformation

$$c_{i\uparrow}^\dagger \rightarrow c_{i\uparrow}^\dagger (p_3 p_4)^{-\frac{1}{2}n_{i\downarrow}} \quad c_{i\downarrow}^\dagger \rightarrow c_{i\downarrow}^\dagger (p_3 p_4)^{\frac{1}{2}n_{i\uparrow}}$$

which yields the following local Hamiltonians:

$$\begin{aligned} h_{i(i+1)} = & -p_2^{-1} p_3 p_4^{-1} c_{i\uparrow}^\dagger c_{(i+1)\uparrow} (q^{-1} p_3^{-1} p_4 [\alpha + 1] [\alpha]^{-1})^{\frac{1}{2}n_{i\downarrow}} \\ & \times (q p_3^{-1} p_4 [\alpha + 1] [\alpha]^{-1})^{\frac{1}{2}n_{(i+1)\downarrow}} + \text{h.c.} \\ & - p_1^{-1} p_3^{-1} p_4 c_{i\downarrow}^\dagger c_{(i+1)\downarrow} (q p_3 p_4^{-1} [\alpha + 1] [\alpha]^{-1})^{\frac{1}{2}n_{i\uparrow}} \\ & \times (q^{-1} p_3 p_4^{-1} [\alpha + 1] [\alpha]^{-1})^{\frac{1}{2}n_{(i+1)\uparrow}} + \text{h.c.} \\ & + [\alpha]^{-1} (p_1^{-1} p_2^{-1} c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger c_{(i+1)\downarrow} c_{(i+1)\uparrow} + \text{h.c.}) + [\alpha]^{-1} (n_{i\uparrow} n_{i\downarrow} + n_{i\uparrow} n_{i\downarrow}) \\ & + q^{\alpha+1} (n_{i\uparrow} + n_{i\downarrow} - 1) + q^{-\alpha-1} (n_{(i+1)\uparrow} + n_{(i+1)\downarrow} - 1). \end{aligned} \quad (25)$$

It is clear that there are only three independent gauge parameters p_1 , p_2 , $p_3 p_4^{-1}$. This result is confirmed by the transfer matrix eigenvalue expression and Bethe ansatz equations, which will be derived below, as they exhibit dependency on only these three independent gauge parameters. In the context of our earlier discussion, the parameter $p_3 p_4$ enters into the Hamiltonian by means of a symmetric twistor and thus may be transformed away.

A final comment here is that the above Hamiltonian is not the most general that can be obtained through this procedure. By taking the tensor product representation of $U_q(sl(2|1))$ with different values of the free parameter α in (20) a more general solution of the Yang–Baxter equation is obtained. For the gauge-free case the explicit form is given in [36, 37]. Using the method proposed in [45], a Hamiltonian can still be derived through this solution.

8. Bethe ansatz solution

Having derived the model, we now use the method of the algebraic Bethe ansatz in order to find the eigenvalues of the transfer matrix which in turn permits us to calculate a formula for the energy levels of the Hamiltonian. Since we are working in a ‘higher spin’ representation of $U_q(sl(2|1))$, we follow the approach proposed for the study of the integrable spin 1 chain [33–35]. Rather than seek ‘creation operators’ over the pseudo-vacuum from the action of $R(u)$ (see e.g. [38]) we instead appeal to an algebraic structure derived from a lower rank object. In order to achieve this goal, we first introduce the matrices $\mathcal{R}(u) \in \text{End}(W \otimes W)$ and $L(u) \in \text{End}(W \otimes V)$ where W denotes a three-dimensional space and V is the four-dimensional representation space of $U_q(sl(2|1))$ as before. We require these operators to satisfy the following forms of the Yang–Baxter equation:

$$\mathcal{R}_{12}(u-v)\mathcal{R}_{13}(u)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u)\mathcal{R}_{12}(u-v) \quad (26)$$

$$\mathcal{R}_{12}(u-v)L_{13}(u)L_{23}(v) = L_{23}(v)L_{13}(u)\mathcal{R}_{12}(u-v) \quad (27)$$

$$L_{12}(u-v)L_{13}(u)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)L_{13}(u)L_{12}(u-v) \quad (28)$$

where $R(u)$ is as before with elements given by (21). Explicitly, $\mathcal{R}(u)$ and $L(u)$ satisfying the above relations take the form

$$\mathcal{R}(u) = \begin{pmatrix} \mathcal{R}_{11}^{11} & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 0 & 0 \\ 0 & \mathcal{R}_{12}^{12} & 0 & | & \mathcal{R}_{21}^{12} & 0 & 0 & | & 0 & 0 & 0 \\ 0 & 0 & \mathcal{R}_{13}^{13} & | & 0 & 0 & 0 & | & \mathcal{R}_{31}^{13} & 0 & 0 \\ - & - & - & | & - & - & - & | & - & - & - \\ 0 & \mathcal{R}_{12}^{21} & 0 & | & \mathcal{R}_{21}^{21} & 0 & 0 & | & 0 & 0 & 0 \\ 0 & 0 & 0 & | & 0 & \mathcal{R}_{22}^{22} & 0 & | & 0 & 0 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & \mathcal{R}_{23}^{23} & | & 0 & \mathcal{R}_{32}^{23} & 0 \\ - & - & - & | & - & - & - & | & - & - & - \\ 0 & 0 & \mathcal{R}_{13}^{31} & | & 0 & 0 & 0 & | & \mathcal{R}_{31}^{31} & 0 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & \mathcal{R}_{23}^{32} & | & 0 & \mathcal{R}_{32}^{32} & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 0 & \mathcal{R}_{33}^{33} \end{pmatrix}$$

$$\begin{aligned} \mathcal{R}_{11}^{11} &= [u-1] & \mathcal{R}_{12}^{12} &= p_1^{-1}p_2p_3^{-1}p_4[u] \\ \mathcal{R}_{13}^{13} &= p_2p_3^{-1}p_4[u] & \mathcal{R}_{21}^{12} &= -q^u \\ * \mathcal{R}_{31}^{13} &= q^u & \mathcal{R}_{21}^{21} &= p_1p_2^{-1}p_3p_4^{-1}[u] \\ \mathcal{R}_{22}^{22} &= [u-1] & \mathcal{R}_{23}^{23} &= p_1p_3p_4^{-1}[u] \\ \mathcal{R}_{12}^{21} &= -q^{-u} & * \mathcal{R}_{32}^{23} &= q^u \\ \mathcal{R}_{31}^{31} &= p_2^{-1}p_3p_4^{-1}[u] & \mathcal{R}_{32}^{32} &= p_1^{-1}p_3^{-1}p_4[u] \\ * \mathcal{R}_{33}^{33} &= [u+1] & \mathcal{R}_{13}^{31} &= -q^{-u} \\ \mathcal{R}_{23}^{32} &= -q^{-u} \end{aligned} \quad (29)$$

$$L(u) = \begin{pmatrix} L_{11}^{11} & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & L_{12}^{12} & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & L_{13}^{13} & 0 & | & 0 & L_{22}^{13} & 0 & 0 & | & L_{31}^{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{14}^{14} & | & 0 & 0 & 0 & 0 & | & 0 & L_{32}^{14} & 0 & 0 \\ - & - & - & - & | & - & - & - & - & | & - & - & - & - \\ 0 & 0 & 0 & 0 & | & L_{21}^{21} & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & L_{13}^{22} & 0 & | & 0 & L_{22}^{22} & 0 & 0 & | & L_{31}^{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & L_{23}^{23} & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & L_{24}^{24} & | & 0 & 0 & L_{33}^{24} & 0 \\ - & - & - & - & | & - & - & - & - & | & - & - & - & - \\ 0 & 0 & L_{13}^{31} & 0 & | & 0 & L_{22}^{31} & 0 & 0 & | & L_{31}^{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{14}^{32} & | & 0 & 0 & 0 & 0 & | & 0 & L_{32}^{32} & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & L_{24}^{33} & | & 0 & 0 & L_{33}^{33} & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & L_{34}^{34} \end{pmatrix}$$

$$\begin{aligned} L_{11}^{11} &= p_1^{-1}[u - \alpha/2 - 1] & L_{12}^{12} &= [u - \alpha/2 - 1] \\ L_{13}^{13} &= p_1^{-1}p_2p_3^{-1}p_4[u - \alpha/2] & L_{14}^{14} &= p_2p_3^{-1}p_4[u - \alpha/2] \\ *L_{31}^{13} &= -q^{u+1/2}p_1^{-1}p_4\sqrt{[\alpha]} & L_{22}^{13} &= -q^{u-\alpha/2} \\ *L_{32}^{14} &= q^u\sqrt{[\alpha+1]} & L_{21}^{21} &= p_2^{-1}[u - \alpha/2 - 1] \\ L_{22}^{22} &= p_1p_2^{-1}p_3p_4^{-1}[u - \alpha/2] & L_{23}^{23} &= [u - \alpha/2 - 1] \\ L_{24}^{24} &= p_1p_3p_4^{-1}[u - \alpha/2] & L_{13}^{22} &= -q^{-u+\alpha/2} \\ *L_{31}^{22} &= q^{u-1/2}p_2^{-1}p_3\sqrt{[\alpha]} & *L_{33}^{24} &= q^u\sqrt{[\alpha+1]} \\ L_{31}^{31} &= p_1^{-1}p_2^{-1}[u + \alpha/2 - 1] & *L_{32}^{32} &= p_2^{-1}p_3p_4^{-1}[u + \alpha/2] \\ *L_{33}^{33} &= p_1^{-1}p_3^{-1}p_4[u + \alpha/2] & L_{34}^{34} &= [u + \alpha/2 + 1] \\ L_{13}^{31} &= q^{-u-1/2}p_1^{-1}p_3^{-1}\sqrt{[\alpha]} & L_{22}^{31} &= -q^{-u+1/2}p_2^{-1}p_4^{-1}\sqrt{[\alpha]} \\ *L_{14}^{32} &= -q^{-u}\sqrt{[\alpha+1]} & *L_{24}^{33} &= -q^{-u}\sqrt{[\alpha+1]}. \end{aligned} \tag{30}$$

The existence of these solutions is a consequence of the representation theory of the untwisted affine extension of $U_q(sl(2|1))$ (e.g. see [30]) where the three-dimensional space corresponds to the fundamental module. Again, multiplication of tensor products in (26)–(28) is to be undertaken with consideration to (2). In order to simplify the task of the Bethe ansatz approach we work here with the solutions of (26)–(28) without the \mathbb{Z}_2 -graded tensor rule. This is achieved by a change of sign by those matrix elements of $\mathcal{R}(u)$, $L(u)$, $R(u)$ prefixed with a $*$. Again, we refer to [30] for a justification of this approach. We comment that in the space W the \mathbb{Z}_2 -grading of the basis states is $(1) = (2) = 0$, $(3) = 1$.

Let us introduce two associative algebras Y and Z with elements $\{Y_j^i(u)\}_{i,j=1}^3$ and $\{Z_j^i(u)\}_{i,j=1}^4$ respectively, and subject to the relations

$$\mathcal{R}_{12}(u - v)Y_{13}(u)Y_{23}(v) = Y_{23}(v)Y_{13}(u)\mathcal{R}_{12}(u - v) \tag{31}$$

$$R_{12}(u - v)Z_{13}(u)Z_{23}(v) = Z_{23}(v)Z_{13}(u)R_{12}(u - v) \tag{32}$$

$$L_{12}(u - v)Y_{13}(u)Z_{23}(v) = Z_{23}(v)Y_{13}(u)L_{12}(u - v) \tag{33}$$

where

$$Y(u) = \sum_{i,j} e_j^i \otimes Y_j^i(u) \quad Z(u) = \sum_{i,j} e_j^i \otimes Z_j^i(u). \tag{34}$$

The associativity of these algebras is guaranteed by the relations equation (1) and (26). By comparison with equation (1) we see that the monodromy matrix provides a representation of the Z algebra acting on the module $V^{\otimes L}$ by the mapping

$$\pi(Z_j^i(v))_l^k = (-1)^{(j)(k)+(j)(l)+(i)(k)} T_{jl}^{ik}(v). \tag{35}$$

Moreover, the transfer matrix is expressible in terms of this representation by

$$t(v)_l^k = \sum_{i=1}^4 (-1)^{(i)+(i)(l)} \pi(Z_i^i(v))_l^k. \quad (36)$$

The phase factors present above are required since the Z is defined in terms of the non-graded L -operator.

We also define an auxiliary monodromy matrix through

$$U(u) = L_{0L}(u) \dots L_{02}(u)L_{01}(u).$$

Likewise, the auxiliary monodromy matrix $U(u)$ gives a representation of the Y algebra through the action

$$\pi(Y_j^i(u))_l^k = (-1)^{(j)(k)+(j)(l)+(i)(k)} U_{jl}^{ik}(u) \quad (37)$$

while the existence of the solution for (28) permits us to appeal to (33) for algebraic relations between elements of Y and Z . In the following we omit the symbol π for ease of notation.

We choose as a reference state (pseudo-vacuum) for the Bethe ansatz procedure the L -fold copy of the highest weight state in V ; namely

$$\Phi^0 = (v^1)^{\otimes L} \quad (38)$$

which itself is an eigenstate of the transfer matrix with the eigenvalue

$$\left(\frac{[v-\alpha]}{[v+\alpha]} \right)^L - \left(p_1^{-1} \frac{[v]}{[v+\alpha]} \right)^L - \left(p_2^{-1} \frac{[v]}{[v+\alpha]} \right)^L + \left(p_1^{-1} p_2^{-1} \frac{[v][v-1]}{[v+\alpha][v-\alpha-1]} \right)^L. \quad (39)$$

In order to obtain more eigenstates of the transfer matrix we adopt the ansatz

$$\Phi^j = \sum_{\beta} S^{\{\beta\}}(\{u\}) \Phi^0 F_{\{\beta\}}^j \quad (40)$$

where the $F_{\{\beta\}}^j$ are undetermined co-efficients and we have set

$$S^{\{\beta\}}(\{u\}) = Y_3^{\beta_1}(u_1) Y_3^{\beta_2}(u_2) \dots Y_3^{\beta_N}(u_N) \quad \beta_i = 1, 2. \quad (41)$$

The motivation for this choice of ansatz is standard in the sense that a routine calculation shows

$$Y_i^j(u) \Phi^0 = 0 \quad \forall j \neq i \neq 3 \quad (42)$$

while Φ^0 is an eigenstate of $Y_i^i(u)$, $i = 1, 2, 3$.

We now appeal to the algebraic relations given by equation (33) in order to determine the constraints on the variables u_i needed to force equation (40) to be an eigenstate. Of the numerous relations resulting from (33) we need only consider the following:

$$\begin{aligned} Z_1^1(v) Y_3^1(u) &= p_2 \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2-1]} Y_3^1(u) Z_1^1(v) - q^{(u-v)} \frac{\sqrt{[\alpha]}}{[u-v-\alpha/2-1]} \\ &\quad \times (q^{1/2} p_2 p_4 Z_3^1(v) Y_1^1(u) - q^{-1/2} p_1 p_3 Z_2^1(v) Y_2^1(u)) \\ Z_1^1(v) Y_3^2(u) &= p_1 \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2-1]} Y_3^2(u) Z_1^1(v) - q^{(u-v)} \frac{\sqrt{[\alpha]}}{[u-v-\alpha/2-1]} \\ &\quad \times (q^{1/2} p_2 p_4 Z_3^1(v) Y_1^2(u) - q^{-1/2} p_1 p_3 Z_2^1(v) Y_2^2(u)) \\ Z_4^4(v) Y_3^1(u) &= p_2 p_3^{-1} p_4 \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2]} Y_3^1(u) Z_4^4(v) \\ &\quad - q^{(u-v)} p_2 p_3^{-1} p_4 \frac{\sqrt{[\alpha+1]}}{[u-v+\alpha/2]} Z_4^2(v) Y_3^3(u) \end{aligned}$$

$$\begin{aligned}
Z_4^4(v)Y_3^2(u) &= p_1 p_3 p_4^{-1} \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2]} Y_3^2(u) Z_4^4(v) \\
&\quad - q^{(u-v)} p_1 p_3 p_4^{-1} \frac{\sqrt{[\alpha+1]}}{[u-v+\alpha/2]} Z_4^3(v) Y_3^3(u) \\
Z_2^2(v)Y_3^1(u) &= -p_2 p_3^{-1} p_4 \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2]} Y_3^1(u) Z_2^2(v) \\
&\quad - q^{(u-v)} p_2 p_3^{-1} p_4 \frac{\sqrt{[\alpha+1]}}{[u-v+\alpha/2]} Z_4^2(v) Y_1^1(u) \\
Z_3^3(v)Y_3^2(u) &= -p_1 p_3 p_4^{-1} \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2]} Y_3^2(u) Z_3^3(v) \\
&\quad - q^{(u-v)} p_1 p_3 p_4^{-1} \frac{\sqrt{[\alpha+1]}}{[u-v+\alpha/2]} Z_4^3(v) Y_2^2(u) \\
Z_3^2(v)Y_3^1(u) &= -p_1 p_3 p_4^{-1} \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2]} Y_3^1(u) Z_3^2(v) \\
&\quad - q^{(u-v)} p_1 p_3 p_4^{-1} \frac{\sqrt{[\alpha+1]}}{[u-v+\alpha/2]} Z_4^2(v) Y_2^1(u) \\
Z_2^3(v)Y_3^2(u) &= -p_2 p_3^{-1} p_4 \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2]} Y_3^2(u) Z_2^3(v) \\
&\quad - q^{(u-v)} p_2 p_3^{-1} p_4 \frac{\sqrt{[\alpha+1]}}{[u-v+\alpha/2]} Z_4^3(v) Y_1^2(u) \\
Z_2^2(v)Y_3^2(u) &= -p_1 \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2-1]} Y_3^2(u) Z_2^2(v) \\
&\quad + p_2 p_3^{-1} p_4 q^{(v-u-\alpha/2)} \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2][u-v+\alpha/2-1]} Y_3^1(u) Z_2^3(v) \\
&\quad + p_2 p_3^{-1} p_4 q^{(u-v)} \frac{\sqrt{[\alpha+1]}}{[u-v+\alpha/2]} Z_4^2(v) Y_1^2(u) \\
&\quad - p_1 p_3 q^{(u-v-1/2)} \frac{\sqrt{[\alpha]}}{[u-v+\alpha/2-1]} Z_2^1(v) Y_3^3(u) \\
&\quad - p_1 p_2 p_4 q^{(2u-2v-1/2)} \frac{\sqrt{[\alpha]}\sqrt{[\alpha+1]}}{[u-v+\alpha/2-1][u-v+\alpha/2]} Z_4^1(v) Y_1^3(u) \\
Z_3^3(v)Y_3^1(u) &= -p_2 \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2-1]} Y_3^1(u) Z_3^3(v) \\
&\quad + p_1 p_3 p_4^{-1} q^{(u-v+\alpha/2)} \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2][u-v+\alpha/2-1]} Y_3^2(u) Z_3^2(v) \\
&\quad + p_1 p_3 p_4^{-1} q^{(u-v)} \frac{\sqrt{[\alpha+1]}}{[u-v+\alpha/2]} Z_4^3(v) Y_2^1(u) \\
&\quad + p_2 p_4 q^{(u-v+1/2)} \frac{\sqrt{[\alpha]}}{[u-v+\alpha/2-1]} Z_3^1(v) Y_3^3(u) \\
&\quad + p_1 p_2 p_3 q^{(2u-2v+1/2)} \frac{\sqrt{[\alpha]}\sqrt{[\alpha+1]}}{[u-v+\alpha/2-1][u-v+\alpha/2]} Z_4^1(v) Y_2^3(u) \\
Z_3^2(v)Y_3^2(u) &= -p_1^2 p_2^{-1} p_3^2 p_4^{-2} \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2-1]} Y_3^2(u) Z_3^2(v)
\end{aligned}$$

$$\begin{aligned}
& + p_1 p_3 p_4^{-1} q^{(v-u-\alpha/2)} \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2][u-v+\alpha/2-1]} Y_3^1(u) Z_3^3(v) \\
& - p_1 p_3 p_4^{-1} q^{(u-v)} \frac{\sqrt{[\alpha+1]}}{[u-v+\alpha/2]} Z_4^2(v) Y_2^2(u) \\
& - p_1 p_3 q^{(u-v-1/2)} \frac{\sqrt{[\alpha]}}{[u-v+\alpha/2-1]} Z_3^1(v) Y_3^3(u) \\
& - p_1^{-2} p_3^2 p_4^{-1} q^{(2u-2v-1/2)} \frac{\sqrt{[\alpha]}\sqrt{[\alpha+1]}}{[u-v+\alpha/2-1][u-v+\alpha/2]} Z_4^1(v) Y_2^3(u) \\
Z_2^3(v) Y_3^1(u) = & - p_1^{-1} p_2^2 p_3^{-2} p_4^2 \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2-1]} Y_3^1(u) Z_2^3(v) \\
& + p_2 p_3^{-1} p_4 q^{(u-v+\alpha/2)} \frac{[u-v-\alpha/2-1]}{[u-v+\alpha/2][u-v+\alpha/2-1]} Y_3^2(u) Z_2^2(v) \\
& - p_2 p_3^{-1} p_4 q^{(u-v)} \frac{\sqrt{[\alpha+1]}}{[u-v+\alpha/2]} Z_4^3(v) Y_1^1(u) \\
& + p_2 p_4 q^{(u-v+1/2)} \frac{\sqrt{[\alpha]}}{[u-v+\alpha/2-1]} Z_2^1(v) Y_3^3(u) \\
& + p_2^{-2} p_3^{-1} p_4^2 q^{(2u-2v+1/2)} \frac{\sqrt{[\alpha]}\sqrt{[\alpha+1]}}{[u-v+\alpha/2-1][u-v+\alpha/2]} Z_4^1(v) Y_1^3(u).
\end{aligned}$$

In the above commutation relations two types of terms arise when the actions of the operators $Z_1^1(v)$, $Z_4^4(v)$ are evaluated on (40). These terms are those giving a vector proportional to Φ^j which we call *wanted terms* while all other types are *unwanted terms* (u.t.). The operators in the above expressions contributing to the unwanted terms are all those of the form $Z_j^i(v) Y_l^k(u)$ with the necessity that $i < j$. Note that for any $Y_l^k(u)$ term which arises where $l \neq 3$ and $k \neq 1, 2$ we can appeal to the commutation relations (31) to commute it through to the reference state Φ^0 where it will act as a scalar. Repeated use of this procedure shows that all unwanted terms are ultimately expressible in the form

$$S^{\{\nu\}}(\{u'\}) Z_j^i(v) S^{\{\nu'\}}(\{u''\}) \Phi^0$$

with $i < j$ and v arbitrary and so can never give a vector proportional to Φ^j .

In terms of the fermionic representation (22), it is deduced that the reference state Φ^0 is the completely filled state with two electrons of opposite spin at each lattice site. Furthermore, it is not difficult to see that the operator $L_3^1(v)$ annihilates a spin-up electron from the system while $L_3^2(v)$ annihilates a spin-down electron. Using the condition that the states (40) have well defined electron number and spin (this can be interpreted as the conservation condition in the sense of Schultz [46]), we have

$$\begin{aligned}
Z_1^1(v) \Phi^j &= p_1^{L-n_-} p_2^{L-n_+} \frac{[v-\alpha]^L}{[v+\alpha]^L} \prod_{i=1}^{2L-N} \frac{[u_i-v-\alpha/2-1]}{[u_i-v+\alpha/2-1]} \Phi^j + \text{u.t.} \\
Z_4^4(v) \Phi^j &= p_1^{-n_-} p_2^{-n_+} p_3^{(n_+-n_-)} p_4^{(n_--n_+)} \frac{[v]^L [v-1]^L}{[v+\alpha]^L [v-\alpha-1]^L} \\
&\quad \times \prod_{i=1}^{2L-N} \frac{[u_i-v-\alpha/2-1]}{[u_i-v+\alpha/2]} \Phi^j + \text{u.t.}
\end{aligned}$$

where n_+ and n_- are respectively the number of spin-up and spin-down electrons in the state and $N = n_+ + n_-$.

The situation for determining the action of $Z_2^2(v)$, $Z_3^3(v)$ is a little more complicated. For these cases more than one wanted term may arise in the calculation, which forces us to employ a nested Bethe ansatz approach. In terms of the six-vertex solution of the Yang–Baxter equation with gauge parameter l

$$r(v, l) = \begin{pmatrix} [v+1] & 0 & 0 & 0 \\ 0 & l[v] & q^v & 0 \\ 0 & q^{-v} & l^{-1}[v] & 0 \\ 0 & 0 & 0 & [v+1] \end{pmatrix} \quad (43)$$

we define the monodromy matrix

$$t(v, \{u\}, l) = r_{01}(v - u_1, l)r_{02}(v - u_2, l) \dots r_{0N}(v - u_N, l).$$

It turns out that in this notation we may write

$$\begin{aligned} Z_2^2(v)\Phi^j &= \left(p_1^{-1} \frac{[v]}{[v+\alpha]} \right)^L \prod_{i=1}^{2L-N} \left(p_2 p_3^{-1} p_4 \frac{[u_i - v - \alpha/2 - 1]}{[u_i - v + \alpha/2][u_i - v + \alpha/2 - 1]} \right) \\ &\quad \times S^{\{\beta^1\}}(\{u\}) t_{1\{\beta^1\}}^{\{\beta^1\}}(v - \alpha/2, \{u\}, p_1 p_2^{-1} p_3 p_4^{-1}) \Phi^0 F_{\{\beta^1\}}^j + \text{u.t.} \\ Z_3^3(v)\Phi^j &= \left(p_2^{-1} \frac{[v]}{[v+\alpha]} \right)^L \prod_{i=1}^{2L-N} \left(p_1 p_3 p_4^{-1} \frac{[u_i - v - \alpha/2 - 1]}{[u_i - v + \alpha/2][u_i - v + \alpha/2 - 1]} \right) \\ &\quad \times S^{\{\beta^1\}}(\{u\}) t_{2\{\beta^1\}}^{\{\beta^1\}}(v - \alpha/2, \{u\}, p_1 p_2^{-1} p_3 p_4^{-1}) \Phi^0 F_{\{\beta^1\}}^j + \text{u.t.} \end{aligned}$$

Completing the diagonalization of the transfer matrix is accomplished by diagonalizing the operator

$$\mu_1 t_1^1(v, \{u\}, l) + \mu_2 t_2^2(v, \{u\}, l) \quad (44)$$

which is the transfer matrix for the six-vertex model with gauge parameter l , inhomogeneities u_i and twisted boundary conditions. This is a standard calculation so we simply present the eigenvalue expression which reads

$$\begin{aligned} \lambda(v) &= \mu_1 l^M \prod_{i=1}^{2L-N} [v - u_i + 1] \prod_{j=1}^M \frac{[v - w_j - 1]}{[v - w_j]} + \mu_2 l^{(M+N-2L)} \prod_{i=1}^{2L-N} [v - u_i] \\ &\quad \times \prod_{j=1}^M \frac{[v - w_j + 1]}{[v - w_j]}. \end{aligned}$$

Making the substitutions

$$\begin{aligned} v &\rightarrow v - \alpha/2 \\ l &\rightarrow p_1 p_2^{-1} p_3 p_4^{-1} \\ \mu_1 &\rightarrow p_1^{-L} p_2^N p_3^{-N} p_4^N \\ \mu_2 &\rightarrow p_1^N p_2^{-L} p_3^N p_4^{-N} \end{aligned}$$

allows us to determine the contribution to the eigenvalue expression for the transfer matrix coming from the terms $Z_2^2(v)$ and $Z_3^3(v)$. At this point we would like to remark that the reference state used in the Bethe ansatz diagonalization of the six-vertex transfer matrix (44) corresponds to the state

$$L_3^1(u_1) L_3^1(u_2) \dots L_3^1(u_N) \Phi^0$$

which is an $2L - N$ electron state containing L spin-down electrons and $L - N$ spin-up electrons. We can immediately deduce that $M = L - n_-$. Collecting these results and making

the substitutions $u_i \rightarrow u_i + 1, w_j \rightarrow w_j + 1/2$, we may now give the eigenvalue expression

$$\begin{aligned}
 \Lambda_1(v, u_i, w_j) &= p_1^{-n_-} p_2^{-n_+} p_3^{(n_+ - n_-)} p_4^{(n_- - n_+)} \frac{[v]^L [v - 1]^L}{[v + \alpha]^L [v - \alpha - 1]^L} \\
 &\times \prod_{i=1}^{2L-N} \frac{[u_i - v - \alpha/2]}{[u_i - v + \alpha/2 + 1]} \\
 &- \left(p_1^{-1} \frac{[v]}{[v + \alpha]} \right)^L \prod_{i=1}^{2L-N} \left(\frac{p_2 p_4 [u_i - v - \alpha/2]}{p_3 [u_i - v + \alpha/2 + 1]} \right) \\
 &\times \prod_{j=1}^{L-n_-} \left(\frac{p_1 p_3 [w_j - v + \alpha/2 + 3/2]}{p_2 p_4 [w_j - v + \alpha/2 + 1/2]} \right) \\
 &- \left(p_2^{-1} \frac{[v]}{[v + \alpha]} \right)^L \prod_{i=1}^{2L-N} \left(p_2 \frac{[u_i - v - \alpha/2]}{[u_i - v + \alpha/2]} \right) \\
 &\times \prod_{j=1}^{L-n_-} \left(\frac{p_1 p_3 [w_j - v + \alpha/2 - 1/2]}{p_2 p_4 [w_j - v + \alpha/2 + 1/2]} \right) \\
 &+ p_1^{L-n_-} p_2^{L-n_+} \frac{[v - \alpha]^L}{[v + \alpha]^L} \prod_{i=1}^{2L-N} \frac{[u_i - v - \alpha/2]}{[u_i - v + \alpha/2]} \tag{45}
 \end{aligned}$$

for the transfer matrix (3). Constraints on the parameters u_i, w_j are given by the Bethe ansatz equations. In principle, these may be derived from the requirement that all the unwanted terms cancel in the above calculation. A more practical way to obtain them is to use the fact that the eigenvalue expression (45) is an analytic function of the variable v . Imposing this condition leads us to the set of equations

$$\begin{aligned}
 p_2^L p_3^{n_- - L} p_4^{L - n_-} \left(\frac{[u_i - \alpha/2]}{[u_i + \alpha/2]} \right)^L &= \prod_{j=1}^{L-n_-} \frac{[u_i - w_j + 1/2]}{[u_i - w_j - 1/2]} \quad i = 1, 2, \dots, 2L - N \\
 p_1^{-L} p_2^L p_3^{N - 2L} p_4^{2L - N} \prod_{i=1}^{2L-N} \frac{[u_i - w_j - 1/2]}{[u_i - w_j + 1/2]} \\
 &= - \prod_{k=1}^{L-n_-} \frac{[w_j - w_k + 1]}{[w_j - w_k - 1]} \quad j = 1, 2, \dots, L - n_-. \tag{46}
 \end{aligned}$$

Alternatively, we can use the following reference state:

$$L_3^2(u_1) L_3^2(u_2) \dots L_3^2(u_N) \Phi^0$$

for the diagonalization of the operator (44). In this instance we obtain

$$\begin{aligned}
 \bar{\Lambda}_1(v, u_i, w_j) &= p_1^{-n_-} p_2^{-n_+} p_3^{(n_+ - n_-)} p_4^{(n_- - n_+)} \frac{[v]^L [v - 1]^L}{[v + \alpha]^L [v - \alpha - 1]^L} \\
 &\times \prod_{i=1}^{2L-N} \frac{[u_i - v - \alpha/2]}{[u_i - v + \alpha/2 + 1]} \\
 &- \left(p_2^{-1} \frac{[v]}{[v + \alpha]} \right)^L \prod_{i=1}^{2L-N} \left(\frac{p_1 p_3 [u_i - v - \alpha/2]}{p_4 [u_i - v + \alpha/2 + 1]} \right) \\
 &\times \prod_{j=1}^{L-n_+} \left(\frac{p_2 p_4 [w_j - v + \alpha/2 + 3/2]}{p_1 p_3 [w_j - v + \alpha/2 + 1/2]} \right)
 \end{aligned}$$

$$\begin{aligned}
 & - \left(p_1^{-1} \frac{[v]}{[v + \alpha]} \right)^L \prod_{i=1}^{2L-N} \left(p_1 \frac{[u_i - v - \alpha/2]}{[u_i - v + \alpha/2]} \right) \\
 & \times \prod_{j=1}^{L-n_+} \left(\frac{p_2 p_4}{p_1 p_3} \frac{[w_j - v + \alpha/2 - 1/2]}{[w_j - v + \alpha/2 + 1/2]} \right) \\
 & + p_1^{L-n_-} p_2^{L-n_+} \frac{[v - \alpha]^L}{[v + \alpha]^L} \prod_{i=1}^{2L-N} \frac{[u_i - v - \alpha/2]}{[u_i - v + \alpha/2]} \tag{47}
 \end{aligned}$$

for the eigenvalues of the transfer matrix. It is clear that (47) is obtained from (45) by the change of parameters (24) and spin reflection, a consequence of the invariance of the Hamiltonian (23) with respect to this operation. For this case the Bethe ansatz equations assume the form

$$\begin{aligned}
 p_1^L p_3^{L-n_+} p_4^{n_+-L} \left(\frac{[u_i - \alpha/2]}{[u_i + \alpha/2]} \right)^L &= \prod_{j=1}^{L-n_+} \frac{[u_i - w_j + 1/2]}{[u_i - w_j - 1/2]} \quad i = 1, 2, \dots, 2L - N \\
 p_1^L p_2^{-L} p_3^{2L-N} p_4^{N-2L} \prod_{i=1}^{2L-N} \frac{[u_i - w_j - 1/2]}{[u_i - w_j + 1/2]} \\
 &= - \prod_{k=1}^{L-n_+} \frac{[w_j - w_k + 1]}{[w_j - w_k - 1]} \quad j = 1, 2, \dots, L - n_+.
 \end{aligned}$$

It is not clear whether the two eigenvalue expressions presented above, either individually or together, give the entire spectrum for the model. No highest weight theorem for the Bethe states is applicable here due to the fact that the imposition of periodic boundary conditions does not preserve invariance with respect to the action of the elements of $U_q(sl(2|1))$. In the $q \rightarrow 1$ limit the invariance is restored and one can show that the Bethe states are indeed highest weight states similar to the methods used for other $sl(2|1)$ invariant models [15,20]. However the combinatorial arguments given in those works do not extend in any obvious manner for this case and to our knowledge the completeness of the Bethe states for the U model remains an open question.

Using the transfer matrix eigenvalue expressions (45), (47) the energies of the Bethe states in both cases are obtained through

$$\begin{aligned}
 E &= - \frac{q^{\alpha+1} - q^{-\alpha-1}}{2 \ln q} \Lambda^{-1} \left. \frac{d\Lambda}{du} \right|_{u=0} \\
 &= \frac{(q^\alpha + q^{-\alpha})[\alpha + 1]L}{[\alpha]} + \sum_{i=1}^{2L-N} \frac{[\alpha][\alpha + 1]}{[u_i + \alpha/2][u_i - \alpha/2]}.
 \end{aligned}$$

One can alternatively use a tensor product of lowest weight states as the pseudo-vacuum for the Bethe ansatz calculation; namely

$$\Phi^0 = (v^A)^{\otimes L}.$$

This leads to a different form for the transfer matrix eigenvalues and Bethe ansatz equations. The procedure used is analogous to that above, and so we will omit the details of the calculation and just give the final results. The transfer matrix eigenvalues are

$$\begin{aligned}
 \Lambda_2(v, u_i, w_j) &= p_1^{L-n_+} p_2^{L-n_-} \frac{[v]^L [v - 1]^L}{[v + \alpha]^L [v - \alpha - 1]^L} \prod_{i=1}^N \frac{[u_i - v + \alpha/2 + 1/2]}{[u_i - v - \alpha/2 + 1/2]} \\
 & - \left(\frac{p_2 p_4}{p_3} \frac{[v]}{[v - \alpha - 1]} \right)^L \prod_{i=1}^N \left(p_1^{-1} \frac{[u_i - v + \alpha/2 + 1/2]}{[u_i - v - \alpha/2 + 1/2]} \right)
 \end{aligned}$$

$$\begin{aligned}
& \times \prod_{j=1}^{n_+} \left(\frac{p_1 p_3 [w_j - v - \alpha/2 + 1]}{p_2 p_4 [w_j - v - \alpha/2]} \right) \\
& - \left(\frac{p_1 p_3 [v]}{p_4 [v - \alpha - 1]} \right)^L \prod_{i=1}^N \left(\frac{p_4 [u_i - v + \alpha/2 + 1/2]}{p_1 p_3 [u_i - v - \alpha/2 - 1/2]} \right) \\
& \times \prod_{j=1}^{n_+} \left(\frac{p_1 p_3 [w_j - v - \alpha/2 - 1]}{p_2 p_4 [w_j - v - \alpha/2]} \right) \\
& + p_1^{-n_+} p_2^{-n_-} p_3^{-(n_- - n_+)} p_4^{(n_+ - n_-)} \frac{[v + \alpha + 1]^L}{[v - \alpha - 1]^L} \prod_{i=1}^N \frac{[u_i - v + \alpha/2 + 1/2]}{[u_i - v - \alpha/2 - 1/2]} \quad (48)
\end{aligned}$$

where the Bethe ansatz equations read

$$\begin{aligned}
p_1^L p_3^{L-n_+} p_4^{n_+-L} \left(\frac{[u_i - \alpha/2 - 1/2]}{[u_i + \alpha/2 + 1/2]} \right)^L &= \prod_{j=1}^{n_+} \frac{[u_i - w_j - 1/2]}{[u_i - w_j + 1/2]} \quad i = 1, 2, \dots, N \\
p_1^{-L} p_2^L p_3^{N-2L} p_4^{2L-N} \prod_{i=1}^N \frac{[u_i - w_j - 1/2]}{[u_i - w_j + 1/2]} &= - \prod_{k=1}^{n_+} \frac{[w_j - w_k + 1]}{[w_j - w_k - 1]} \quad j = 1, 2, \dots, n_+.
\end{aligned}$$

Just as in the previous case there is a second expression for the eigenvalues, due to a different choice of reference state in the nesting, which is

$$\begin{aligned}
\bar{\Lambda}_2(v, u_i, w_j) &= p_1^{L-n_+} p_2^{L-n_-} \frac{[v]^L [v - 1]^L}{[v + \alpha]^L [v - \alpha - 1]^L} \prod_{i=1}^N \frac{[u_i - v + \alpha/2 + 1/2]}{[u_i - v - \alpha/2 + 1/2]} \\
& - \left(\frac{p_1 p_3}{p_4} \frac{[v]}{[v - \alpha - 1]} \right)^L \prod_{i=1}^N \left(p_2^{-1} \frac{[u_i - v + \alpha/2 + 1/2]}{[u_i - v - \alpha/2 + 1/2]} \right) \\
& \times \prod_{j=1}^{n_-} \left(\frac{p_2 p_4 [w_j - v - \alpha/2 + 1]}{p_1 p_3 [w_j - v - \alpha/2]} \right) \\
& - \left(\frac{p_2 p_4 [v]}{p_3 [v - \alpha - 1]} \right)^L \prod_{i=1}^N \left(\frac{p_3 [u_i - v + \alpha/2 + 1/2]}{p_2 p_4 [u_i - v - \alpha/2 - 1/2]} \right) \\
& \times \prod_{j=1}^{n_-} \left(\frac{p_2 p_4 [w_j - v - \alpha/2 - 1]}{p_1 p_3 [w_j - v - \alpha/2]} \right) \\
& + p_1^{-n_+} p_2^{-n_-} p_3^{-(n_- - n_+)} p_4^{(n_+ - n_-)} \frac{[v + \alpha + 1]^L}{[v - \alpha - 1]^L} \\
& \times \prod_{i=1}^N \frac{[u_i - v + \alpha/2 + 1/2]}{[u_i - v - \alpha/2 - 1/2]}. \quad (49)
\end{aligned}$$

For this expression, the Bethe ansatz equations are

$$\begin{aligned}
p_2^L p_3^{n_- - L} p_4^{L - n_-} \left(\frac{[u_i - \alpha/2 - 1/2]}{[u_i + \alpha/2 + 1/2]} \right)^L &= \prod_{j=1}^{n_-} \frac{[u_i - w_j - 1/2]}{[u_i - w_j + 1/2]} \quad i = 1, 2, \dots, N \\
p_1^L p_2^{-L} p_3^{2L-N} p_4^{N-2L} \prod_{i=1}^N \frac{[u_i - w_j - 1/2]}{[u_i - w_j + 1/2]} &= - \prod_{k=1}^{n_-} \frac{[w_j - w_k + 1]}{[w_j - w_k - 1]} \quad j = 1, 2, \dots, n_-.
\end{aligned}$$

One can see that the formulae $\Lambda_2(v, u_i, w_j)$ and $\bar{\Lambda}_2(v, u_i, w_j)$ are related through the mapping (24) and spin reflection as in the previous example. Moreover, there is a mapping from $\Lambda_1(v, u_i, w_j)$ to $\Lambda_2(v, u_i, w_j)$ and $\bar{\Lambda}_1(v, u_i, w_j)$ to $\bar{\Lambda}_2(v, u_i, w_j)$ (and the corresponding Bethe ansatz equations) which is given by

$$\begin{aligned} p_1 &\rightarrow p_2^{-1} p_3 p_4^{-1} & p_2 &\rightarrow p_1^{-1} p_3^{-1} p_4 & p_3 &\rightarrow p_4 \\ p_4 &\rightarrow p_3 & \alpha &\rightarrow -\alpha - 1 & n_+ &\rightarrow L - n_- & n_- &\rightarrow L - n_+. \end{aligned} \quad (50)$$

The existence of this relationship can be understood in terms of the dual representation of (20). Recall that for any representation of a quantum superalgebra π the dual representation π^* is defined for each element a in the superalgebra through

$$\pi^*(a) = \pi(S(a))^{\text{st}}$$

where st denotes the supertransposition as before and S is the antipode. For the representation (20) under consideration here, the dual representation is equivalent (i.e. up to a basis transformation) to the change of variable $\alpha \rightarrow -\alpha - 1$. The change in the gauge parameters is also a result of the dualization of the Cartan elements in the definition of (8).

The transfer matrix eigenvalue expressions (48), (49) for this second form of the Bethe ansatz give the energies of the Bethe states through

$$\begin{aligned} E &= -\frac{q^{\alpha+1} - q^{-\alpha-1}}{2 \ln q} \Lambda^{-1} \frac{d\Lambda}{du} \Big|_{u=0} \\ &= -(q^{\alpha+1} + q^{-\alpha-1})L - \sum_{i=1}^N \frac{[\alpha + 1]^2}{[u_i + \alpha/2 + 1/2][u_i - \alpha/2 - 1/2]}. \end{aligned}$$

A third form of the Bethe ansatz also exists for this model, which was discovered by Pfanmüller and Frahm [18] in the isotropic case ($q = 1$) with all gauge parameters equal to 1. The extension to the present model is achieved by a similar construction and so we will again omit the details and refer the interested reader to [18] for the methodology. Again, there are two types of eigenvalue expression which are

$$\begin{aligned} \Lambda_3(v, u_i, w_j) &= \left(\frac{p_2[v]}{[v + \alpha]} \right)^L \prod_{i=1}^{L-n_-} \frac{p_1[u_i - v - \alpha/2 + 1/2]}{[u_i - v + \alpha/2 + 1/2]} \prod_{j=1}^{n_+} \frac{[w_j - v + \alpha/2]}{p_2[w_j - v - \alpha/2]} \\ &+ \left(\frac{p_4[v]}{p_1 p_3 [v - \alpha - 1]} \right)^L \prod_{i=1}^{L-n_-} \left(\frac{p_1 p_3 [u_i - v - \alpha/2 - 1/2]}{p_4 [u_i - v + \alpha/2 + 1/2]} \right) \\ &\times \prod_{j=1}^{n_+} \left(\frac{p_3 [w_j - v + \alpha/2 + 1]}{p_2 p_4 [w_j - v - \alpha/2]} \right) \\ &- \left(\frac{p_2 p_4 [v]^2}{p_1 p_3 [v + \alpha] [v - \alpha - 1]} \right)^L \prod_{i=1}^{L-n_-} \left(\frac{p_1 [u_i - v - \alpha/2 + 1/2]}{[u_i - v + \alpha/2 + 1/2]} \right) \\ &\times \prod_{j=1}^{n_+} \left(\frac{p_3 [w_j - v + \alpha/2 + 1]}{p_2 p_4 [w_j - v - \alpha/2]} \right) \\ &- \prod_{i=1}^{L-n_-} \frac{p_1 p_3 [u_i - v - \alpha/2 - 1/2]}{p_4 [u_i - v + \alpha/2 + 1/2]} \prod_{j=1}^{n_+} \frac{[w_j - v + \alpha/2]}{p_2 [w_j - v - \alpha/2]} \end{aligned} \quad (51)$$

subject to the Bethe ansatz equations

$$\begin{aligned} p_1^{-L} p_3^{n_+-L} p_4^{L-n_+} \left(\frac{[u_i + \alpha/2 + 1/2]}{[u_i - \alpha/2 - 1/2]} \right)^L &= \prod_{j=1}^{n_+} \frac{[w_j - u_i - 1/2]}{[w_j - u_i + 1/2]} \\ p_2^L p_3^{n_+-L} p_4^{L-n_+} \left(\frac{[w_j - \alpha/2]}{[w_j + \alpha/2]} \right)^L &= \prod_{i=1}^{L-n_+} \frac{[u_i - w_j - 1/2]}{[u_i - w_j + 1/2]} \end{aligned} \quad (52)$$

for which the energy is given by

$$E = \sum_{i=1}^{L-n_+} \frac{[\alpha + 1]^2}{[u_i + \alpha/2 + 1/2][u_i - \alpha/2 - 1/2]} - \sum_{j=1}^{n_+} \frac{[\alpha][\alpha + 1]}{[w_j + \alpha/2][w_j - \alpha/2]}$$

and

$$\begin{aligned} \bar{\Lambda}_3(v, u_i, w_j) &= \left(\frac{p_1[v]}{[v + \alpha]} \right)^L \prod_{i=1}^{L-n_+} \frac{p_2[u_i - v - \alpha/2 + 1/2]}{[u_i - v + \alpha/2 + 1/2]} \prod_{j=1}^{n_+} \frac{[w_j - v + \alpha/2]}{p_1[w_j - v - \alpha/2]} \\ &+ \left(\frac{p_3[v]}{p_2 p_4 [v - \alpha - 1]} \right)^L \prod_{i=1}^{L-n_+} \left(\frac{p_2 p_4 [u_i - v - \alpha/2 - 1/2]}{p_3 [u_i - v + \alpha/2 + 1/2]} \right) \\ &\times \prod_{j=1}^{n_+} \left(\frac{p_4 [w_j - v + \alpha/2 + 1]}{p_1 p_3 [w_j - v - \alpha/2]} \right) \\ &- \left(\frac{p_1 p_3 [v]^2}{p_2 p_4 [v + \alpha][v - \alpha - 1]} \right)^L \prod_{i=1}^{L-n_+} \left(\frac{p_2 [u_i - v - \alpha/2 + 1/2]}{[u_i - v + \alpha/2 + 1/2]} \right) \\ &\times \prod_{j=1}^{n_+} \left(\frac{p_4 [w_j - v + \alpha/2 + 1]}{p_1 p_3 [w_j - v - \alpha/2]} \right) \\ &- \prod_{i=1}^{L-n_+} \frac{p_2 p_4 [u_i - v - \alpha/2 - 1/2]}{p_3 [u_i - v + \alpha/2 + 1/2]} \prod_{j=1}^{n_+} \frac{[w_j - v + \alpha/2]}{p_1 [w_j - v - \alpha/2]} \end{aligned} \quad (53)$$

in which case the Bethe ansatz equations are

$$\begin{aligned} p_2^{-L} p_3^{L-n_+} p_4^{n_+-L} \left(\frac{[u_i + \alpha/2 + 1/2]}{[u_i - \alpha/2 - 1/2]} \right)^L &= \prod_{j=1}^{n_+} \frac{[w_j - u_i - 1/2]}{[w_j - u_i + 1/2]} \\ p_2^L p_3^{n_+-L} p_4^{L-n_+} \left(\frac{[w_j - \alpha/2]}{[w_j + \alpha/2]} \right)^L &= \prod_{i=1}^{L-n_+} \frac{[u_i - w_j - 1/2]}{[u_i - w_j + 1/2]} \end{aligned} \quad (54)$$

and the energy expression is

$$E = \sum_{i=1}^{L-n_+} \frac{[\alpha + 1]^2}{[u_i + \alpha/2 + 1/2][u_i - \alpha/2 - 1/2]} - \sum_{j=1}^{n_+} \frac{[\alpha][\alpha + 1]}{[w_j + \alpha/2][w_j - \alpha/2]}.$$

Again, the two eigenvalue expressions and associated Bethe ansatz equations are related through spin reflection coupled with the change of parameters (24). A more striking feature is that (51)–(54) are all invariant under (50) with an accompanying interchange of the sets of parameters $\{u_i\}$ and $\{v_i\}$.

9. Equivalence of the transfer matrix eigenvalues

The above calculations show that there are three different forms for the Bethe ansatz solution of this model. Here we will argue that these forms are in fact equivalent. The method that

we will use is inspired by the paper [18] which hints at the argument we will give without providing many details. Our goal here is to make the argument more transparent.

We begin with the following R -matrix [29, 30] which is associated with the quantum superalgebra $U_q(sl(1|1))$. The solution we present below is more general than that of [29, 30] in that we have included a gauge parameter l which is necessary for our argument to succeed.

The R -matrix is

$$r(v, l)^{\alpha\beta} = \begin{pmatrix} [v - (\alpha + \beta)/2 - 1] & 0 & 0 & 0 \\ 0 & l[v + (\alpha - \beta)/2] & q^v \sqrt{[\alpha + 1][\beta + 1]} & 0 \\ 0 & -q^{-v} \sqrt{[\alpha + 1][\beta + 1]} & l^{-1}[v + (\beta - \alpha)/2] & 0 \\ 0 & 0 & 0 & [v + (\alpha + \beta)/2 + 1] \end{pmatrix} \quad (55)$$

which satisfies the coloured Yang–Baxter equation

$$r_{12}^{\alpha\beta}(u - v, l)r_{13}^{\alpha\gamma}(u, l)r_{23}^{\beta\gamma}(v, l) = r_{23}^{\beta\gamma}(v, l)r_{13}^{\alpha\gamma}(u, l)r_{12}^{\alpha\beta}(u - v, l).$$

We emphasize again that the tensor products in the above equation are to be evaluated according to the rule (2). In (55) we have adopted the convention that index 1 is odd and 2 is even.

Using this solution we can construct the transfer matrix

$$\tau(v, \alpha, \beta, u_i, l, x_1, x_2) = \text{str}_0(X_0 r_{01}^{\alpha 0}(v - u_1, l) \dots r_{0P}^{\alpha 0}(v - u_P, l) \\ \times r_{0(P+1)}^{\alpha\beta}(v, l) \dots r_{0(L+P)}^{\alpha\beta}(v, l))$$

where $X = \text{diag}(x_1, x_2)$, $x \in \mathbb{C}$. These matrices form a commuting family with the property

$$[\tau(v, \alpha, \beta, u_i, l, x_1, x_2), \tau(u, \alpha, \gamma, u_i, l, x_1, x_2)] = 0.$$

For the diagonalization of the above transfer matrix by the Bethe ansatz there are two available approaches. We may begin with a reference state given by

$$(v^2)^{\otimes(L+P)}$$

in which case the eigenvalues read

$$\lambda^-(v, \alpha, \beta, u_i, w_j, l, x_1, x_2) = \left(x_2 [v + (\alpha + \beta)/2 + 1]^L \prod_{i=1}^P [v - u_i + \alpha/2 + 1] \right. \\ \left. - x_1 l^L [v + (\alpha - \beta)/2]^L \prod_{i=1}^P l [v - u_i + \alpha/2] \right) \\ \times \prod_{j=1}^M \frac{[w_j - v + \alpha/2 + 1/2]}{l [w_j - v - \alpha/2 - 1/2]} \quad (56)$$

such that the parameters w_j satisfy the Bethe ansatz equations

$$x_2 \left(\frac{[w_j + \beta/2 + 1/2]}{[w_j - \beta/2 - 1/2]} \right)^L = x_1 l^L \prod_{i=1}^P \frac{l [u_i - w_j + 1/2]}{[u_i - w_j - 1/2]}. \quad (57)$$

We can also use

$$(v^1)^{\otimes(L+P)}$$

for the pseudo-vacuum which yields the eigenvalue expression

$$\lambda^+(v, \alpha, \beta, u_i, \bar{w}_j, l, x_1, x_2) = - \left(x_1 [v - (\alpha + \beta)/2 - 1]^L \prod_{i=1}^P [v - u_i - \alpha/2 - 1] \right. \\ \left. - x_2 l^{-L} [v + (\beta - \alpha)/2]^L \prod_{i=1}^P l^{-1} [v - u_i - \alpha/2] \right) \\ \times \prod_{j=1}^{L+P-M-1} \frac{l [\bar{w}_j - v - \alpha/2 - 1/2]}{[\bar{w}_j - v + \alpha/2 + 1/2]} \quad (58)$$

such that the parameters \bar{w}_j satisfy the Bethe ansatz equations

$$x_2 \left(\frac{[\bar{w}_j + \beta/2 + 1/2]}{[\bar{w}_j - \beta/2 - 1/2]} \right)^L = x_1 L^L \prod_{i=1}^P \frac{L[u_i - \bar{w}_j + 1/2]}{[u_i - \bar{w}_j - 1/2]}. \tag{59}$$

We conjecture that for each set of M parameters $\{w_j\}$ satisfying (57), there is a set of $(L + P - M - 1)$ parameters $\{\bar{w}_i\}$ satisfying (59) which render (56) and (58) equal. In the limit $p_i = q \rightarrow 1$ this argument can be made rigorous since in this instance there is an underlying $sl(1|1)$ invariance for the transfer matrix. As mentioned before, one can prove a highest and lowest weight theorem respectively for the two sets of Bethe states and then use a combinatorial argument to claim that the multiplets generated by the Bethe states give the full space of states in each case, so there must be a one-to-one correspondence. Although this proof will fail for generic values of q , we still believe that the result holds true.

Applying the conjecture, it then follows that

$$\begin{aligned} \Lambda_3(v, u_i, w_j) &= \frac{1}{[v + \alpha]^L} \prod_{j=1}^{L-n_-} \frac{1}{[v - u_i - \alpha/2 - 1/2]} \\ &\times \left\{ -\lambda^- \left(v, \alpha - 1, \alpha - 1, u_i, w_j, p_2, \left(\frac{p_1}{p_2} \right)^{(L-n_-)}, \left(\frac{p_1 p_3}{p_4} \right)^{(L-n_-)} \right) \right. \\ &+ \frac{[v]^L}{[v - \alpha - 1]^L} \lambda^- \left(v - 1/2, \alpha, \alpha - 1, u_i, w_j, \right. \\ &\left. \left. \frac{p_2 p_4}{p_3}, p_1^{-n_-} \left(\frac{p_2 p_4}{p_3} \right)^{(n_- - L)}, \left(\frac{p_1 p_3}{p_4} \right)^{-n_-} \right) \right\} \\ &= \frac{1}{[v + \alpha]^L} \prod_{j=1}^{L-n_-} \frac{1}{[v - u_i - \alpha/2 - 1/2]} \\ &\times \left\{ -\lambda^+ \left(v, \alpha - 1, \alpha - 1, u_i, \bar{w}_j, p_2, \left(\frac{p_1}{p_2} \right)^{(L-n_-)}, \right. \right. \\ &\left. \left. \left(\frac{p_1 p_3}{p_4} \right)^{(L-n_-)}, \left(\frac{p_1 p_3}{p_4} \right)^{-n_-} \right) \right. \\ &+ \frac{[v]^L}{[v - \alpha - 1]^L} \lambda^+ \left(v - 1/2, \alpha, \alpha - 1, u_i, \bar{w}_j, \right. \\ &\left. \left. \frac{p_2 p_4}{p_3}, p_1^{-n_-} \left(\frac{p_2 p_4}{p_3} \right)^{(n_- - L)}, \left(\frac{p_1 p_3}{p_4} \right)^{-n_-} \right) \right\} \tag{60} \\ &= \Lambda_1(v, \bar{w}_j, u_i). \tag{61} \end{aligned}$$

Similarly, we find

$$\begin{aligned} \bar{\Lambda}_3(v, u_i, w_j) & \tag{62} \\ &= \frac{1}{[v + \alpha]^L} \prod_{j=1}^{L-n_+} \frac{1}{[v - u_i - \alpha/2 - 1/2]} \\ &\times \left\{ -\lambda^- \left(v, \alpha - 1, \alpha - 1, u_i, w_j, p_1, \left(\frac{p_2}{p_1} \right)^{(L-n_+)}, \left(\frac{p_2 p_4}{p_3} \right)^{(L-n_+)} \right) \right. \\ &+ \frac{[v]^L}{[v - \alpha - 1]^L} \lambda^- \left(v - 1/2, \alpha, \alpha - 1, u_i, w_j, \right. \end{aligned}$$

$$\begin{aligned} & \left. \frac{p_1 p_3}{p_4}, p_2^{-n_+} \left(\frac{p_1 p_3}{p_4} \right)^{(n_+-L)}, \left(\frac{p_2 p_4}{p_3} \right)^{-n_+} \right\} \\ = & \frac{1}{[v + \alpha]^L} \prod_{j=1}^{L-n_+} \frac{1}{[v - u_i - \alpha/2 - 1/2]} \\ & \times \left\{ -\lambda^+ \left(v, \alpha - 1, \alpha - 1, u_i, \bar{w}_j, p_1, \left(\frac{p_2}{p_1} \right)^{(L-n_+)} \right. \right. \\ & \left. \left. \left(\frac{p_2 p_4}{p_3} \right)^{(L-n_+)}, \left(\frac{p_2 p_4}{p_3} \right)^{-n_+} \right) \right. \\ & \left. + \frac{[v]^L}{[v - \alpha - 1]^L} \lambda^+ \left(v - 1/2, \alpha, \alpha - 1, u_i, \bar{w}_j, \frac{p_1 p_3}{p_4}, p_2^{-n_+} \right. \right. \\ & \left. \left. \left(\frac{p_1 p_3}{p_4} \right)^{(n_+-L)}, \left(\frac{p_2 p_4}{p_3} \right)^{-n_+} \right) \right\} \tag{63} \\ = & \bar{\Lambda}_1(v, \bar{w}_j, u_i) \tag{64} \end{aligned}$$

although the result is immediate through use of (24). In a similar fashion it can be shown that

$$\Lambda_3(v, u_i, w_j) = \Lambda_2(v, \bar{u}_i, w_j) \quad \bar{\Lambda}_3(v, u_i, w_j) = \bar{\Lambda}_2(v, \bar{u}_i, w_j)$$

though the judicious mathematician would appeal to (50) to obtain the result. It is important to note finally that although the eigenvalue expressions are equivalent, the three different Bethe ansatz approaches will not produce the same set of *eigenvectors*. More specifically, for a given transfer matrix eigenvalue the three alternate eigenvectors will be characterized by different quantum numbers n_+ , n_- . This feature can be understood in terms that the Bethe ansatz alone does not generally give a complete set of states for the system as in the examples [15, 20].

10. Quantum transfer matrix eigenvalues

The Bethe ansatz approach for the diagonalization of the quantum transfer matrix proceeds in much the same way as the previous discussion. We return to the algebraic relations (31)–(33). A representation of the algebras Y and Z are obtained through

$$\begin{aligned} \pi(Y(v)) &= (L_{\mathcal{L}0}^{\text{st}\mathcal{L}}(\chi - v)L_{0(\mathcal{L}-1)}(\chi + v) \dots L_{20}^{\text{st}2}(\chi - v)L_{01}(\chi + v)) \\ \pi(Z(v)) &= (R_{\mathcal{L}0}^{\text{st}\mathcal{L}}(\chi - v)R_{0(\mathcal{L}-1)}(\chi + v) \dots R_{20}^{\text{st}2}(\chi - v)R_{01}(\chi + v)) \end{aligned}$$

where the label 0 refers to the auxiliary space (three-dimensional for $L(v)$ and four-dimensional for $R(v)$) and the natural numbers label quantum spaces. The quantum transfer matrix is given by

$$Q(v) = \sum_{i=1}^4 (-1)^{(i)+(i)(k)} \pi(Z_i^i(v))_l^k$$

in complete analogy with (36). The principle difference is in the choice of the reference state. In order to adopt the ansatz (40) we need to find a reference state which satisfies the conditions (42). This is achieved with the choice

$$\Phi^0 = (v^1 \otimes v^4)^{\otimes \mathcal{L}/2}$$

which is an eigenstate of the quantum transfer matrix with eigenvalue

$$\begin{aligned} & \left(\frac{[\chi + v - \alpha][\chi - v][\chi - v - 1]}{p_1 p_2 [\chi + v + \alpha][\chi - v + \alpha][\chi - v - \alpha - 1]} \right)^{\mathcal{L}/2} \\ & - \left(\frac{p_3 [\chi + v][\chi - v]}{p_1 p_2 p_4 [\chi + v + \alpha][\chi - v - \alpha - 1]} \right)^{\mathcal{L}/2} \\ & - \left(\frac{p_4 [\chi + v][\chi - v]}{p_1 p_2 p_3 [\chi + v + \alpha][\chi - v - \alpha - 1]} \right)^{\mathcal{L}/2} \\ & + \left(\frac{[\chi + v][\chi + v - 1][\chi - v + \alpha + 1]}{p_1 p_2 [\chi + v + \alpha][\chi + v - \alpha - 1][\chi - v - \alpha - 1]} \right)^{\mathcal{L}/2}. \end{aligned}$$

At this point we can follow the Bethe ansatz procedure exactly as before. This yields the eigenvalue expression for the quantum transfer matrix

$$\begin{aligned} \Lambda(v) &= (p_1 p_2^{-1} p_3^2 p_4^{-2})^{\mathcal{M}} \left(\frac{[\chi + v][\chi + v - 1][\chi - v + \alpha + 1]}{p_1 p_2 [\chi + v + \alpha][\chi + v - \alpha - 1][\chi - v - \alpha - 1]} \right)^{\mathcal{L}/2} \\ & \times \prod_{i=1}^{\mathcal{N}} \frac{p_2 p_4 [u_i - v - \alpha/2]}{p_3 [u_i - v + \alpha/2 + 1]} \\ & \times \left(\frac{p_3 [\chi + v][\chi - v]}{p_1 p_2 p_4 [\chi + v + \alpha][\chi - v - \alpha - 1]} \right)^{\mathcal{L}/2} \prod_{i=1}^{\mathcal{N}} \left(\frac{p_2 p_4 [u_i - v - \alpha/2]}{p_3 [u_i - v + \alpha/2 + 1]} \right) \\ & \times \prod_{j=1}^{\mathcal{M}} \left(\frac{p_1 p_3 [w_j - v + \alpha/2 + 3/2]}{p_2 p_4 [w_j - v + \alpha/2 + 1/2]} \right) \\ & - \left(\frac{p_4 [\chi + v][\chi - v]}{p_1 p_2 p_3 [\chi + v + \alpha][\chi - v - \alpha - 1]} \right)^{\mathcal{L}/2} \prod_{i=1}^{\mathcal{N}} \left(\frac{p_2 [u_i - v - \alpha/2]}{[u_i - v + \alpha/2]} \right) \\ & \times \prod_{j=1}^{\mathcal{M}} \left(\frac{p_1 p_3 [w_j - v + \alpha/2 - 1/2]}{p_2 p_4 [w_j - v + \alpha/2 + 1/2]} \right) \\ & + (p_1 p_2^{-1})^{-\mathcal{M}} \left(\frac{[\chi + v - \alpha][\chi - v][\chi - v - 1]}{p_1 p_2 [\chi + v + \alpha][\chi - v + \alpha][\chi - v - \alpha - 1]} \right)^{\mathcal{L}/2} \\ & \times \prod_{i=1}^{\mathcal{N}} \frac{p_2 [u_i - v - \alpha/2]}{[u_i - v + \alpha/2]} \end{aligned}$$

subject to the Bethe ansatz equations

$$\begin{aligned} & \left(\frac{p_3 [\chi + u_i - \alpha/2][\chi - u_i - \alpha/2 - 1]}{p_4 [\chi + u_i + \alpha/2][\chi - u_i + \alpha/2]} \right)^{\mathcal{L}/2} = \prod_{j=1}^{\mathcal{M}} \frac{p_3 [w_j - u_i - 1/2]}{p_4 [w_j - u_i + 1/2]} \\ & \left(\frac{p_4}{p_3} \right)^{\mathcal{L}} \prod_{i=1}^{\mathcal{N}} \frac{p_3 [u_i - w_j + 1/2]}{p_4 [u_i - w_j - 1/2]} = - \prod_{k=1}^{\mathcal{M}} \frac{[w_j - w_k - 1]}{[w_j - w_k + 1]}. \end{aligned}$$

Another expression is obtained through (24). It is curious that the gauge parameters p_1 , p_2 do not appear in these Bethe ansatz equations. As explained earlier, equivalent forms, which we will not make explicit here, can be obtained through the use of the equivalence of the expressions (56), (58).

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